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A Functional Equation with Conjugate Means Derived from a Weighted Arithmetic Mean

Annop Sonubon and *Somsak Orankitjaroen

Department of Mathematics, Faculty of Science
Mahidol University, Bangkok, Thailand

E-mail: murfy_pm@hotmail.com and somsak.ora@mahidol.ac.th

*Corresponding author

ABSTRACT

In this paper, we seek a solution of a functional equation with conjugate means derived from a weighted arithmetic mean; that is, finding continuous strictly monotonic functions φ and ψ on an open interval I which is a solution of

$$\varphi^{-1}\left(p\varphi(x)+q\varphi(y)+(1-p-q)\varphi(tx+(1-t)y)\right)$$
$$+\psi^{-1}\big(r\psi(x)+s\psi(y)+(1-r-s)\psi(tx+(1-t)y)\big)=x+y,$$

for all $x,y\in I$ where $p,q,r,s,t\in(0,1)$, $p\neq q$, $r\neq s$, $p+q\neq 1$, $r+s\neq 1$, $st=r(1-t)$ with either the conditions $p+q=r+s$ or $p+q=2(r+s)$. We found that the solutions φ and ψ are in the form of linear functions.

Keywords: Mean, functional equation, conjugate mean.

1. INTRODUCTION

The question of equality of means has long been investigated since its genesis in the work of Matkowski-Sutô type problem (Matkowski (1999); Sutô (1914)). Since then, many researchers have undertaken to examine such a question for various types of means with certain regularity assumptions. Among those are in the works of Jarczyk (2007), Losonczi (1999), Losonczi and Páles (2011), Makó and Páles (2008), Matkowski (2011) and Páles (2011). With the advent of the class of the conjugate means introduced by Daróczy and Páles (2005), Burai and Dascăl (2012) examined the conjugate means derived from the arithmetic mean; introduction therein contains a summary of other cases that has been done earlier. In the same spirit with the work of Burai and Dascăl (2012), we extend their study to the conjugate means derived from the weighted arithmetic mean. To clarify what we intend to study, we use the following notations: denoting I as an open interval in $\mathbb{R}$, we let $\mathcal{CM}(I)$ be the class of continuous and strictly monotone real valued functions on I. Suppose that $L: I \times I \to I$ is a strict mean on I. A function $M: I \times I \to I$ is called the *conjugate mean derived from the mean* L on I if there exist $p, q \in (0,1]$ and $\varphi \in \mathcal{CM}(I)$ such that

$$M(x,y) = \varphi^{-1}(p\varphi(x) + q\varphi(y) + (1-p-q)\varphi(L(x,y))) := L_{\varphi}^{(p,q)}(x,y)$$

for all $x, y \in I$. We call p and q the *weights* and φ the *generating function* of the mean M. It can be shown that M is also a strict mean on I. If $p+q=1$, the mean M is reduced to the well-known weighted quasi-arithmetic mean. To our interest here we consider the conjugate mean derived from a weighted arithmetic mean $A(x,y) := tx + (1-t)y$ for the weight $t \in (0,1)$.

The intended study mentioned at the beginning can be formulated as in the question: for what weights $p, q, r, s, t \in (0,1)$ and generating functions $\varphi, \psi \in \mathcal{CM}(I)$ will the arithmetic mean of two conjugate means $A_{\varphi}^{(p,q)}$ and $A_{\psi}^{(r,s)}$ be equal to the arithmetic mean itself, that is

$$A_{\varphi}^{(p,q)}(x,y) + A_{\psi}^{(r,s)}(x,y) = x + y \quad \text{for all } x, y \in I. \tag{1}$$

In our result, to avoid degenerate cases, we assume that $p \neq q, r \neq s$, $p+q \neq 1, r+s \neq 1$. We also make further assumptions that $st = r(1-t)$ with either the conditions $p+q = r+s$ or $p+q = 2(r+s)$. Let us proceed with the necessary theorem, and propositions needed to solve our problem.

2. PRELIMINARIES

Two functions $\varphi, \psi \in \mathcal{CM}(I)$ are said to be equivalent on I if there exist real constants a, b with $a \neq 0$ such that $\psi(x) = a\varphi(x) + b$ for all $x \in I$ written as $\psi(x) \sim \varphi(x)$ for all $x \in I$. A function φ satisfies $\mathcal{C}^n$ if φ is $n-$ times continuously differentiable with non-zero first order derivatives. To solve the invariance equation (1) we adopt the notation of the generalization of quasi-arithmetic means introduced by Makó and Páles (2009) and proceed as follows.

Consider a signed measure μ on $[0,1]$; the kth moment, $\hat{\mu}_k$, and the kth centralized moment, μ_k, of μ are given by

$$\hat{\mu}_k := \int_0^1 \tau^k \, d\mu(\tau) \text{ and } \mu_k := \int_0^1 (\tau - \hat{\mu}_1)^k \, d\mu(\tau)$$

where k is a nonnegative integer. Obviously, $\hat{\mu}_0 = \mu_0 = 1$and $\mu_1 = 0$. Let δ_a denote the Dirac measure concentrated at the point $a \in [0,1]$. Let $\varphi \in \mathcal{CM}(I)$. Then our conjugate mean $A_\varphi^{(p,q)}$ can be expressed as

$$A_\varphi^{(p,q)}(x,y) = \varphi^{-1}\left(\int_0^1 \varphi(\tau x + (1-\tau)y) \, d\mu(\tau)\right)$$

where $\mu = q\delta_0 + p\delta_1 + (1-p-q)\delta_t$ and $p, q, t \in (0,1)$. The formula for $A_\psi^{(r,s)}$ is similar. It follows that the first moment $\hat{\mu}_1 = p + (1-p-q)t$, where as $\hat{\nu}_1 = r + (1-r-s)t$. With $\hat{\mu}_1$ and $\hat{\nu}_1$, the k th centralized moment μ_k, ν_k are

$$\begin{aligned} \mu_k = (-1)^k q\big((1-q)t + p(1-t)\big)^k + p\big(qt + (1-p)(1-t)\big)^k \\ +(1-p-q)(qt - p(1-t))^k \end{aligned} \tag{2}$$

$$\begin{aligned} \nu_k = (-1)^k s\big((1-s)t + r(1-t)\big)^k + r\big(st + (1-r)(1-t)\big)^k \\ +(1-r-s)\big(st - r(1-t)\big)^k. \end{aligned} \tag{3}$$

Our main result is an application of the following Theorem 2.1 and Propositions 2.2-2.3 from Makó and Páles (2009). Note that the assumptions of Borel probability measure can be extended naturally to signed measure.

Theorem 2.1. Let μ and ν be a Borel probability measures with $\mu_2\nu_2 \neq 0$ and satisfying $(\mu_3, \nu_3) \neq 3(\hat{\mu}_1 - \hat{\nu}_1)(-\mu_2^2, \nu_2^2)/(\mu_2 + \nu_2)$. Assume also that φ, ψ satisfy $\mathcal{C}^3$. Then the equation $L_\varphi^{(p,q)}(x,y) + L_\psi^{(r,s)}(x,y) = x + y$ holds if and only if $\hat{\mu}_1 + \hat{\nu}_1 = 1$ and

(i) either there exist real constants a, b, c, d with $ac \neq 0$ such that
$$\varphi(x) = ax + b \quad \text{and} \quad \psi(x) = cx + d \ \ (x \in I),$$

(ii) or there exist real constants $a, b, c, d, \alpha, \beta$ with $ac \neq 0, \alpha\beta < 0$ such that

$$\varphi(x) = ae^{\alpha x} + b \quad \text{and} \quad \psi(x) = ce^{\beta x} + d \ \ (x \in I),$$

and for $n \in \mathbb{N}$,

$$\sum_{i=0}^{n} \binom{n}{i} \alpha^i \beta^{n-i} (\mu_{i+1}\nu_{n-i} + \mu_i \nu_{n+1-i}) = 0,$$

(iii) or there exist real constants $a, b, c, d, \alpha, \beta$ with $ac \neq 0, (\alpha - 1)(\beta - 1) < 0$ and $x_0 \notin I$ such that, for $x \in I$,

$$\varphi(x) = \begin{cases} a|x - x_0|^{\alpha} + b & \text{if } \alpha \neq 0, \\ a\ln|x - x_0| + b & \text{if } \alpha = 0 \end{cases}$$

and

$$\psi(x) = \begin{cases} c|x - x_0|^{\beta} + d & \text{if } \beta \neq 0, \\ c\ln|x - x_0| + d & \text{if } \beta = 0. \end{cases}$$

Proposition 2.2. Let μ and ν be a Borel probability measures with $\mu_2\nu_2 \neq 0$. If $\alpha\beta \neq 0$ and there exists a solution of the equation $L_{\varphi}^{(p,q)}(x, y) + L_{\psi}^{(r,s)}(x, y) = x + y$ of the form (ii) in Theorem 2.1, then $\alpha/\beta = -\nu_2 / \mu_2$ and the following condition must be valid

$$\sum_{i=0}^{n} (-1)^i \binom{n}{i} \frac{\mu_{i+1}\nu_{n-i} + \mu_i \nu_{n+1-i}}{\mu_2^i \nu_2^{n-i}} \quad (n \in \mathbb{N}). \tag{4}$$

Proposition 2.3. Let μ and ν be a Borel probability measures with $\mu_2\nu_2 \neq 0$ and $(\mu_3, \nu_3) \neq 3(\hat{\mu}_1 - \hat{\nu}_1)(-\mu_2^2, \nu_2^2)/(\mu_2 + \nu_2)$. If $(\alpha - 1)(\beta - 1) < 0$ and there exists a solution of the equation $L_{\varphi}^{(p,q)}(x, y) + L_{\psi}^{(r,s)}(x, y) = x + y$ of the form (iii) in Theorem 2.1, then $\mu_3\nu_2^2 + \nu_3\mu_2^2 \neq 0$,

$$\alpha = (2\mu_3\nu_2^2 + \nu_3\mu_2(\mu_2 - \nu_2) + 3(\hat{\mu}_1 - \hat{\nu}_1)\mu_2\nu_2^2)/(\mu_3\nu_2^2 + \nu_3\mu_2^2),$$
$$\beta = (2\nu_3\mu_2^2 + \mu_3\nu_2(\nu_2 - \mu_2) + 3(\hat{\nu}_1 - \hat{\mu}_1)\mu_2^2\nu_2)/(\mu_3\nu_2^2 + \nu_3\mu_2^2)$$

and the following condition must hold

$$\begin{gathered} 27\mu_2^3\nu_2^3(\hat{\mu}_1 - \hat{\nu}_1)^2(\mu_2 - \nu_2) + 6\mu_2^2\nu_2^2(\nu_2 - \mu_2)\mu_3\nu_3 \\ +18\mu_2^2\nu_2^2(\mu_2 - \nu_2)(\hat{\mu}_1 - \hat{\nu}_1)(\mu_3\nu_2 - \nu_3\mu_2) \\ +(-12\nu_2^2(\hat{\mu}_1 - \hat{\nu}_1)^2 + 3\mu_2\nu_2(\mu_2 - \nu_2) + 8(\hat{\mu}_1 - \hat{\nu}_1)(\mu_2 + \nu_2)\nu_3)\nu_2^2\mu_3^2 \\ -(-12\mu_2^2(\hat{\mu}_1 - \hat{\nu}_1)^2 + 3\mu_2\nu_2(\nu_2 - \mu_2) + 8(\hat{\nu}_1 - \hat{\mu}_1)(\mu_2 + \nu_2)\mu_3)\mu_2^2\nu_3^2 \end{gathered}$$

$$+\left(3\nu_2^2(\hat{\nu}_1-\hat{\mu}_1)+\nu_3(\mu_2+\nu_2)\right)(-3\hat{\mu}_1\mu_2+3\mu_2\hat{\nu}_1+\mu_3)\mu_4\nu_2^2 \tag{5}$$
$$+\left(3\nu_2^2(\hat{\nu}_1-\hat{\mu}_1)+\nu_3(\mu_2+\nu_2)\right)(\mu_2\nu_2+2\mu_2^2)\mu_4\nu_3$$
$$-\left(3\mu_2^2(\hat{\mu}_1-\hat{\nu}_1)+\mu_3(\mu_2+\nu_2)\right)(3\hat{\mu}_1\nu_2-3\hat{\nu}_1\nu_2+\nu_3)\nu_4\mu_2^2$$
$$-\left(3\mu_2^2(\hat{\mu}_1-\hat{\nu}_1)+\mu_3(\mu_2+\nu_2)\right)(\mu_2\nu_2+2\nu_2^2)\nu_4\mu_3=0.$$

3. MAIN RESULTS

It is enough to solve our problem up to the equivalence of the functions φ and ψ. We assume that $\varphi'(x)>0$ and $\psi'(x)>0$ if $x\in I$. In the following lemma, we give the first necessary condition for the invariance equation of the conjugate means.

Lemma 3.1. Let $p,q,r,s,t\in(0,1)$ and φ,ψ satisfy $\mathcal{C}^1$ such that the invariance equation (1) holds for all $x,y\in I$, then

$$(p+(1-p-q)t)+(r+(1-r-s)t)=1. \tag{6}$$

Proof. Differentiate both sides of (1) with respect to x and replace y by x yield the result. The above result can be restated in terms of the first moments of μ and ν as $\hat{\mu}_1+\hat{\nu}_1=1$ since

$$\hat{\mu}_1=p+(1-p-q)t,\ \hat{\nu}_1=r+(1-r-s)t. \tag{7}$$

Next lemma expresses the relationship between the solutions φ and ψ.

Lemma 3.2. Assume $p,q,r,s,t\in(0,1)$. If φ,ψ satisfy $\mathcal{C}^2$ and the invariance equation (1), then there exists a positive constant c such that

$$\varphi'(x)=c\psi'(x)^{-\nu_2/\mu_2} \quad \text{for all} \quad x\in I.$$

Proof. Twice differentiate both sides of (1) with respect to x and replace y by x yield

$$\mu_2\frac{\varphi''}{\varphi'}+\nu_2\frac{\psi''}{\psi'}=0, \tag{8}$$

where

$$\mu_2=(p+(1-p-q)t^2)-(p+(1-p-q)t)^2$$

and

$$\nu_2 = (r + (1 - r - s)t^2) - (r + (1 - r - s)t)^2.$$

Integrating (8) gives the result.

It can be easily shown that μ_2 and ν_2 are positive. Lemma 3.1 indicates that if we fix the values of p, q, r and s, then t will change according to (6). With the same assumptions of Lemma 3.1, if we further assume that $st = r(1 - t)$, then p, q, r, s and t are related to the formula

$$\frac{r}{r + s} = \frac{1 - p}{2 - p - q} \tag{9}$$

which can also be expressed as

$$(1 - p)(1 - t) = (1 - q)t. \tag{10}$$

At this point we have equations at our disposal to compute the centralized moments μ_k and ν_k necessary for our result. The centralized moments will be given in the next two lemmas. To shorten our computations hereafter, we write $T = t(1 - t)$, $P = p + q$ and $R = r + s$.

Lemma 3.3. Let $p, q, r, s, t \in (0,1)$ and φ, ψ satisfy $\mathcal{C}^1$ and the invariance equation (1). If $st = r(1 - t)$, then the five centralized moments of μ and ν are

$$\mu_1 = 0 \qquad \mu_2 = TP \qquad \mu_3 = T(1 - 2t)(1 - 2P)$$
$$\mu_4 = T\big(T(8 - 11P) + (3P - 2)\big)$$
$$\mu_5 = T(1 - 2t)\big(T(13P - 11) + (3 - 4P)\big);$$
$$\nu_1 = 0 \qquad \nu_2 = TR \qquad \nu_3 = TR(1 - 2t)$$
$$\nu_4 = TR(1 - 3T) \qquad \nu_5 = TR(1 - 2t)(1 - 2T).$$

Proof. The assumption of $st = r(1 - t)$ allows us to reduce the formula (2) of μ_k to $\mu_k = (-1)^k q(1 - t)^k + pt^k + (1 - p - q)(2t - 1)^k$ for $k \geq 1$. Evidently, $\mu_0 = 1$. With the help of (9), we have

$$\begin{aligned}\mu_1 &= -q(1 - t) + pt + (1 - p - q)(2t - 1)\\ &= t(2 - p - q) - (1 - p) = 0.\end{aligned}$$

Straightforward computations of μ_2 with (10) lead to

$\mu_2 = q(1-t)^2 + pt^2 + (1-p-q)(2t-1)^2 = t(1-t)(p+q) = TP.$
The calculations of μ_3, μ_4 and μ_5 are similar, so do the computations of the five centralized moments of ν with the help of formula (3).

Note here that with the assumption $st = r(1-t)$, the first moment $\hat{\mu}_1 = 1-t$ and $\hat{\nu}_1 = t$. We are now in a position to prove the main result:

Theorem 3.4. Let $p, q, r, s, t \in (0,1), r \neq s, p \neq q$ and $p+q \neq 1$, $r+s \neq 1$. If $\varphi, \psi \in \mathcal{CM}(I)$ satisfy $\mathcal{C}^3$ and the invariance equation (1), $st = r(1-t)$, and either $p+q = r+s$ or $p+q = 2(r+s)$, then

$$\varphi(x) \sim x, \ \ \psi(x) \sim x \quad \text{for all } x \in I.$$

Proof. Let φ, ψ satisfy $\mathcal{C}^3$ and the invariance equation (1); they thus satisfy (8) in Lemma 3.2. Rearranging (8) and introducing a new function ϕ by

$$\phi := \mu_2 \frac{\varphi''}{\varphi'} = -\nu_2 \frac{\psi''}{\psi'}\ , \tag{11}$$

we can derive a necessary condition for the validity of the invariance equation (1) in terms of the first order differential equation of ϕ. This is due to Theorem 7, p.14 of Makó and Páles (2009) when applying for the conjugate means derived from the weighted arithmetic mean. Such the first order differential equation of ϕ reads as

$$\left(\frac{3\hat{\mu}_1\mu_2 + \mu_3}{\mu_2} - \frac{3\hat{\nu}_1\nu_2 + \nu_3}{\nu_2}\right)\phi' + \left(\frac{\mu_3}{\mu_2^2} + \frac{\nu_3}{\nu_2^2}\right)\phi^2 = 0. \tag{12}$$

First let assume $st = r(1-t)$and $p+q = r+s$. To solve equation (12) we break our analysis in 3 cases.

Case 1. $\phi = 0$. In this case, we clearly see that this is a trivial solution of (12). As a result, we have $\varphi'' = \psi'' = 0$, and thus φ, ψ are linear functions. It can be checked easily that such a form of φ, ψ is always a solution of the invariance equation (1). Actually the assumptions of $st = r(1-t)$ and $p+q = r+s$ can be dropped in this case.

For the other two cases, we may assume that ϕ is not identically zero. Consequently, there exists a maximal subinterval J of I where ϕ differs from zero. It is evident that J is open and nonempty. In the interval J, (12) can be rewritten as

$$\frac{\phi'}{\phi^2} = -\frac{\mu_3 \nu_2^2 + \nu_3 \mu_2^2}{\mu_2 \nu_2 (\nu_2 (3\hat{\mu}_1 \mu_2 + \mu_3) - \mu_2 (3\hat{\nu}_1 \nu_2 + \nu_3))} . \tag{13}$$

The two cases are those where the right-hand side of (13) is zero and nonzero.

Here we claim that the denominator of the right-hand side of (13) does not vanish. To verify our claim, first note that μ_2, ν_2 are positive as a consequence of Lemma 3.3. Next we can express another factor of the denominator as $\nu_2 (3\hat{\mu}_1 \mu_2 + \mu_3) - \mu_2 (3\hat{\nu}_1 \nu_2 + \nu_3) = T^2 R(1-2t)$ which clearly different from zero by our assumptions. Thus, the denominator does not vanish on J. Now we consider the case when the right-hand side of (13) is zero in J.

Case 2. $\phi' / \phi^2 = 0$. This case means that the numerator of the right-hand side of (13) equals to zero. Using the centralized moments μ_2, μ_3 and ν_2, ν_3 obtained from Lemma 3.3, we have

$$\mu_3 \nu_2^2 + \nu_3 \mu_2^2 = T^3 R(1-2t)\big((P-R)^2 + R(1-R)\big).$$

That is

$$T^3 R(1-2t)\big((P-R)^2 + R(1-R)\big) = 0. \tag{14}$$

Clearly, $T^3 R(1-2t) \neq 0$ by the main hypothesis. The assumption $P = R$ thus implies $R = 1$; that is $r + s = 1$ which is a contradiction. Therefore, case 2 cannot occur. However, this case is exactly a necessary condition (4), when $n = 2$, in Proposition 2.2 for the existence of solutions in the equivalent form of exponential functions or form (ii) of Theorem 2.1.

The last case is the one when the right-hand side is nonzero in J.

Case 3. $\phi' / \phi^2 = c \neq 0$. Setting $\alpha = 1 + (\mu_2 c)^{-1}$, we observe that α cannot be 1. Integrating the considered equation for ϕ yields

$$\frac{1}{\phi(x)} = \frac{x - x_0}{\mu_2(\alpha - 1)} \tag{15}$$

for all $x \in J$. It is evident from the solution that $x_0 \notin J$. It can be shown that $J = I$. Substituting ϕ from (15) into (11), we get

$$\frac{\varphi''(x)}{\varphi'(x)} = \frac{\alpha - 1}{x - x_0}$$

which has a solution in the equivalent form as

$$\varphi(x) \sim \begin{cases} |x - x_0|^\alpha & \text{if} \quad \alpha \neq 0, \\ \ln|x - x_0| & \text{if} \quad \alpha = 0. \end{cases}$$

Using Lemma 3.2, we have

$$\psi(x) \sim \begin{cases} |x - x_0|^\beta & \text{if} \quad \beta \neq 0, \\ \ln|x - x_0| & \text{if} \quad \beta = 0 \end{cases}$$

when $\beta = 1 - (\alpha - 1)\frac{\mu_2}{\nu_2}$. Note $(\alpha - 1)(\beta - 1) = -(\alpha - 1)^2 \frac{\mu_2}{\nu_2} < 0$.

To apply Proposition 2.3, we substitute in (5) the moment $\hat{\mu}_{1,}\mu_2, \mu_3, \mu_4$, $\hat{\nu}_1, \nu_2, \nu_3, \nu_4$ obtained from (7), Lemma 3.3. Simplifying the substitution result, we have

$$\begin{aligned} &P(P - 2R)(2P^2 - 4RP + R)T \\ &-(P^2 - 2RP + P + R)(2P^2 - 4RP + P + 2R)T \\ &+3PRT(P - R) - (1 - P)(P - R)(P - 2R)(1 - 4T) = 0. \end{aligned} \tag{16}$$

The assumption $P = R$ reduces the above equation to $6R^2(R - 1)T = 0$, which is impossible by the assumptions $R > 0, R \neq 1$ and $T > 0$. By the virtue of Proposition 2.3, we conclude that the invariance equation (1) has no solution in the form (iii) of Theorem 2.1.

Next if we assume that $P = 2R$ in place of $P = R$, we can still examine in 3 cases as in the above analysis. In case 1, there is no change in the proof, and we have the solution of (1) in the equivalent form of linear functions. In case 2, upon substituting $P = 2R$ in (14) leads to the conclusion that $R = 0$ which is impossible. Thus, no solution in the equivalent form of exponential functions or form (ii) in Theorem 2.1. In case 3, (16) with $P = 2R$, we have $R = 0$ which is again a contradiction. Consequently, no solution in the form (iii) of Theorem 2.1.

We therefore conclude that the invariance equation with the mentioned assumptions has only the solution in the equivalent form of linear functions according to Theorem 2.1.

4. CONCLUSION

In this paper, we aim to find a solution of the invariance equation (1). With the assumption $p, q, r, s, t \in (0,1),\ p \neq q,\ r \neq s,\ p + q \neq 1,$ $r + s \neq 1,\ st = r(1 - t)$ and either the main conditions

$$p + q = r + s \text{ or } p + q = 2(r + s), \tag{17}$$

we found that the generating functions φ and ψ of the invariance equation are in the form of linear functions. Actually, the main conditions (17) can be dropped for the proof of the nonexistence of solution equivalent to exponential functions.

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